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Similarity solutions of the deformed Maxwell–Bloch system

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Abstract. We have found similarity reductions for the deformed Maxwell–Bloch system to the fifth and second Painlevé equations. Asymptotics of the solutions of these equations, which are relevant to two-level atomic systems with pumping, have also been derived.

1. Introduction

There has been considerable interest in the Painlevé equations over the last fifteen years. The main reason for this interest is the appearance of these equations as similarity reductions of equations that are integrable via the inverse scattering transform, and in the asymptotic analysis of correlation functions of exactly solvable quantum models. Despite the many similarity reductions and asymptotic properties of Painlevé equations which have already been found, in many cases the properties of the similarity solutions of nonlinear soliton-bearing equations are still unknown. The difficulty involved in this approach is usually the complexity of the expressions of similarity solutions in terms of Painlevé equations and, thereby, application of the known asymptotic properties of these equations. This is the problem we address in this paper in the context of a model in nonlinear optics.

We shall consider in this paper the decoupled deformed Maxwell–Bloch system:

$$\begin{aligned} &\partial_\eta E = \varrho \qquad \partial_\eta \bar{E} = \bar{\varrho} \\ &\partial_\xi \varrho = NE \qquad \partial_\xi \bar{\varrho} = N\bar{E} \\ &\partial_\xi N + \tfrac{1}{2}(E\bar{\varrho} + \bar{E}\varrho) = 4s \end{aligned} \tag{1}$$

where ξ, η and s are complex variables and E, $\bar{E}$, ϱ, $\bar{\varrho}$ and N are smooth complex functions of these three independent variables. Note that bars do not necessarily denote complex conjugation.

We have chosen this system because it shares the difficulty mentioned above and because there are two reductions of system (1) which are of physical importance.

The first reduction of system (1), $\bar{E} = E^*$, $\bar{\varrho} = \varrho^*$ and $N = N^*$ with real ξ, η and s (the asterisk means complex conjugation), was reported in [1]. In this reference the possibility of applying the inverse scattering transform with a variable spectral parameter to this system was pointed out, and a possible physical interpretation as a two-level atomic system (Maxwell–Boch system) with a pumping mechanism discussed. We shall treat this case in more detail in section 5.

The second reduction of system (1), $\bar{E} = -E^*$, $\bar{\varrho} = -\varrho^*$ and $N = N^*$ with real ξ, η and s, was reported in [2] in another physical context.

It appears that a correlation function of the one-dimensional impenetrable Bose gas with a δ-function potential, an exactly solvable quantum model, is given by a special solution of (1). The asymptotic behaviour of this solution was thoroughly investigated in [2] for different values of the parameters. In one region of the parameter space the long-distance asymptotics of this special solution shows similarity. In the case of the first reduction mentioned above, the asymptotics of physically relevant solutions have not yet been found.

In the case of the second reduction above and also of the Maxwell–Bloch system without pumping [3], similarity solutions play an important role in the asymptotic behaviour of the physical solutions. As a first step in this direction for system (1) we consider the asymptotic behaviour of its similarity solutions and specify those parts of the parameter space where these asymptotics can describe a real physical process.

The first similarity reduction of system (1) to the fifth Painlevé equation ($\mathbf{P}_5$) was reported by Winternitz in [4] in the particular case when both the dependent and independent variables are real. In this case the reduction was made possible by application of results by Bureau for equations with the Painlevé property and which are quadratic with respect to the second derivative [5]. The asymptotics of the particular case of $\mathbf{P}_5$ which appears in this connection were investigated in [6]. There are two difficulties in applying the results of [6] however. The first difficulty is related to the fact that only the leading term in the asymptotic behaviour of E can be obtained. This term does not depend on any of the parameters of the full solution and, therefore, it is not clear how these parameters should be chosen for the real reduction that is essential in the approach of [4]. The second difficulty is related to finding the correct expression for N. Here one needs to take into account not only the leading but also the next to leading terms in the $\mathbf{P}_5$ asymptotics in the case considered in [6], and these have not yet been obtained.

In order to avoid these difficulties we have applied the method proposed in [7], in which similarity reductions of the Einstein–Maxwell equations, which are also known [8] to be completely integrable via the inverse scattering transform with variable spectral parameter, are dealt with. The basic idea of the method applied in [7] is to specify the data of the Riemann–Hilbert (RH) problem in the plane of the 'hidden' spectral parameter and thereby to deduce a similarity substitution and the auxiliary linear ordinary differential equation with respect to the spectral parameter. It turns out that the similarity solutions themselves define isomonodromic deformations of the auxiliary equation. This makes it possible to apply the methods described in [9, 11, 12] to the evaluation of the asymptotic properties of the similarity solutions. The structure of the paper is as follows. In section 2 we formulate in the usual way a generalized problem. Any solution of this RH problem generates a solution of system (1). In sections 3 and 4 we define specifications of the data of the RH problem such that they correspond to the similarity reductions of (1) to the fifth ($\mathbf{P}_5$) and second ($\mathbf{P}_2$) Painlevé equations respectively. The physical background of the problem is discussed and the appropriate asymptotics of the similarity solutions are reported in section 5.

2. The matrix Riemann–Hilbert problem

Let us consider the linear system

$$\partial_\xi \Psi = \lambda \sigma_3 \Psi + U_0 \Psi \tag{2}$$

$$\partial_\eta \Psi + \frac{s}{\lambda} \partial_\lambda \Psi = \frac{1}{4\lambda} \hat{\varrho} \Psi \tag{3}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{\varrho} = \begin{pmatrix} N & -\varrho \\ -\bar{\varrho} & -N \end{pmatrix} \qquad U_0 = \frac{1}{2}\begin{pmatrix} 0 & E \\ -\bar{E} & 0 \end{pmatrix}$$

and $\Psi(\xi, \eta, \lambda; s)$ is a 2×2 matrix function, and λ is a spectral parameter.

It is easy to prove [1] that the set of equations (1) is the compatibility condition

$$\left[\partial_\xi - \lambda\sigma_3 - U_0, \partial_\eta + \frac{s}{\lambda}\partial_\lambda - \frac{1}{4\lambda}\hat{\varrho}\right] = 0$$

for this system.

Another equivalent formulation of the problem is to allow λ to depend on η, in which case equation (3) will be replaced by

$$\partial_\eta \Psi = \frac{1}{4\lambda}\hat{\varrho}\Psi \tag{4}$$

and the linear system (2) with (4) will be supplemented with the equation for the spectral parameter λ,

$$\partial_\eta \lambda = s/\lambda. \tag{5}$$

Equation (5) can immediately be integrated to give

$$\lambda = \sqrt{2s\eta + k^2} \tag{6}$$

where k^2 is a constant of integration. This observation provides the motivation for introducting into the system (2) with (3) a 'hidden' spectral parameter k in the form

$$k = \sqrt{\lambda^2 - 2s\eta}. \tag{7}$$

The following analytical properties in the complex k-plane of the function Ψ can now easily be deduced from equations (2) and (3):

(i) The function $\Psi(\xi, \eta, k)$ is holomorphic and invertible (in the matrix sense) in the complex k-plane excluding the points $k = \infty$, $k_{0,1} = \pm\sqrt{-2s\eta}$, and probably a set of points $\{a_i\}$ and oriented contours $\{\Gamma_j\}$ which are both independent of ξ and η.

(ii) At $k = \infty$ the matrix function Ψ has an essential singularity and in its vicinity it behaves asymptotically as

$$\Psi(\xi, \eta, k) = \left(I + \frac{\Psi_{-1}}{k} + \mathcal{O}\left(\frac{1}{k^2}\right)\right)\exp\left((k\xi - \theta_\infty \ln k)\sigma_3\right) \qquad \theta_\infty \in \mathcal{C}. \tag{8}$$

(iii) At the points $k = k_{0,1}$ the matrix function Ψ has a regular singularity and in its vicinity it behaves asymptotically as

$$\Psi(\xi, \eta, k) \sim \sum_{n=0}^{\infty} \Psi_n(\xi, \eta)(k - k_r)^n (k - k_r)^{\theta_r \sigma_3/2} P_r; \theta_r \in \mathcal{C} \qquad r = 0, 1. \tag{9}$$

Here P_r are 2×2 matrices independent of ξ and η with $\det P_r \neq 0$.

(iv) For $k \in \{a_i\}$ both regular and essential singularities are allowed. Leading terms of the corresponding asymptotic expansions do not depend on ξ and η.

(v) $\det \Psi(\xi, \eta, k) = \text{constant} \neq 0$.

The following conjugation problem is posed on the set of contours $\{\Gamma_j\}$:

$$\Psi_+(\xi, \eta, k) = \Psi_-(\xi, \eta, k) G_j(k) \qquad k \in \Gamma_j \tag{10}$$

where $\Psi_\pm(\xi, \eta, k)$ are the boundary values of $\Psi(\xi, \eta, k)$ on Γ_j from the left and from the right respectively, and the conjugation matrices $G_j(k)$ do not depend on ξ and η.

The problem of the construction of the Ψ-function as a solution of the conjugation problem (10) such that it satisfies the five properties above, we shall call hereafter the generalized RH problem. In the following we refer to the set of contours $\{\Gamma_j\}$, the corresponding set of conjugation matrices $\{G_j(k)\}$, the above mentioned singular points with matrices P_r and the constants θ_r and θ_∞ as the data of the RH problem.

Once the RH problem is solved, i.e. the Ψ-function with the desired analytical properties is constructed, then by standard arguments (see e.g. [13]) that make use of Liouville's theorem, it can easily be proven that this Ψ-function satisfies equations (2) and (3). Furthermore, from this solution the functions $E(\xi, \eta)$ and $\bar{E}(\xi, \eta)$ can be found by applying

$$\frac{1}{2}\begin{pmatrix} 0 & E(\xi, \eta) \\ -\bar{E}(\xi, \eta) & 0 \end{pmatrix} = [\Psi_{-1}(\xi, \eta), \sigma_3]. \tag{11}$$

The other components of the solution, namely $\varrho(\xi, \eta)$, $\bar{\varrho}(\xi, \eta)$ and $N(\xi, \eta)$, can be found directly from the nonlinear system (1).

We do not discuss here the complicated general problem of specification of the data of the RH problem in such a way that the RH problem becomes solvable. Instead of this we consider in sections 3 and 4 two particular examples of this specification.

3. Similarity reduction to the fifth Painlevé equation

To begin with let us define the isomonodromic class of the solutions of system (1) by the condition

$$G_j(k) = e^{(P_n(k)+\alpha \ln k)\sigma_3} G_j^{(0)} e^{-(P_n(k)+\alpha \ln k)\sigma_3} \tag{12}$$

where $G_j(k)$ are the conjugation matrices defined in (10) and the elements of matrices $G_j^{(0)}$ are independent of k. The coefficients of polynomials $P_n(k)$ and constant α are independent of ξ, η and j. In other words the independence from j means that the factorization (12) is the same for each of the contours Γ_j. For other completely integrable systems the isomonodromic solutions are defined in an analogous way. Similarity solutions always belong to the isomonodromic class with one (so far) known exception [14]. It is an interesting problem to find the spectral interpretation of this case.

Let us consider the simplest particular case of representation (12) when the G_j are constant matrices independent of k. Using now the properties of the Ψ-function (see the previous section), it is easy to prove that the expession

$$(k\partial_k \Psi + 2\eta \partial_\eta \Psi - \xi \partial_\xi \Psi) \cdot \Psi^{-1} \tag{13}$$

is a bounded entire analytical function in the complex k-plane. Liouville's theorem now tells us that expression (13) is a constant which can easily be deduced from the asymptotic expansion (8).

In this way we find an additional equation,

$$k\Psi_k + 2\eta\Psi_\eta = \xi\Psi_\xi - \theta_\infty\sigma_3\Psi \tag{14}$$

for function Ψ. This equation can be solved by the method of characteristics:

$$\frac{dk}{k} = \frac{d\eta}{2\eta} = \frac{d\xi}{-\xi}. \tag{15}$$

The two integrals of equation (15), namely

$$\mu = k\eta^{-1/2} \qquad z = \xi\eta^{1/2} \tag{16}$$

lead to the following form for function Ψ,

$$\Psi(\xi, \eta, k) = \exp(-\tfrac{1}{2}\theta_\infty \ln\eta\,\sigma_3)\Phi(\mu, z). \tag{17}$$

In terms of the self-similar substitution (16) with (17), the asymptotic expansion (8) can be rewritten for $\mu \sim \infty$ as

$$\Phi(\mu, z) = \left(I + \eta^{\theta_\infty\sigma_3/2-1/2}\left(\frac{\Psi_{-1}(\xi, \eta)}{\mu}\right)\eta^{-\theta_\infty\sigma_3/2} + \mathcal{O}\left(\frac{1}{\mu^2}\right)\right)\exp((z\mu - \theta_\infty \ln\mu)\sigma_3). \tag{18}$$

Expansion (18) means that the coefficient of the leading term $\Phi_{-1}(z)$ of the expansion of $\Phi(\mu, z)$ is given by

$$\Psi_{-1}(\xi, \eta) = \eta^{1/2-\theta_\infty\sigma_3/2}\Phi_{-1}(z)\eta^{\theta_\infty\sigma_3/2}. \tag{19}$$

Notice that Φ_{-1} is only a function of the self-similar variable z. A comparison of equations (19) and (11) now leads to a self-similar form for $E(\xi, \eta)$ and $\bar{E}(\xi, \eta)$,

$$E(\xi, \eta) = \eta^{1/2-\theta_\infty}\mathcal{E}(z) \qquad \bar{E}(\xi, \eta) = \eta^{1/2+\theta_\infty}\bar{\mathcal{E}}(z). \tag{20}$$

The corresponding self-similar substitutions for polarization and inversion can be deduced from the nonlinear system (1),

$$\varrho(\xi, \eta) = \eta^{-1/2-\theta_\infty}\rho(z) \qquad \bar{\varrho}(\xi, \eta) = \eta^{-1/2+\theta_\infty}\bar{\rho}(z) \qquad N(\xi, \eta) = \eta^{-1/2}n(z). \tag{21}$$

As mentioned in the introduction, RH problems of type (10) cannot be completely solved. In order to identify a soluble case we should define the data of the RH problem more precisely. To this end in system (2) with (3) we change the variables ξ, η and λ, where λ is given by equation (6), to the self-similar variables μ and z defined in (16).

It is also convenient to shift the regular singularities at $k = k_{0,1}$ to points $k = 0$ and $k = 1$, respectively. In this way we arrive at the transformation

$$\begin{aligned} &\Psi(\xi, \eta, k) = \exp(-\tfrac{1}{4}\tau\sigma_3)\mathcal{Y}(\zeta, \tau) \\ &\tau = 4\sqrt{2s}\xi\eta^{1/2} = 4\sqrt{2s}z \qquad \zeta = (\lambda/2\sqrt{2s\eta} + \tfrac{1}{2}) \end{aligned} \tag{22}$$

under which system (2), (3) takes the form

$$\partial_\zeta \mathcal{Y} = \left(\frac{\tau}{2}\sigma_3 + \frac{\mathcal{V}_0}{\zeta} + \frac{\mathcal{V}_1}{\zeta - 1}\right)\mathcal{Y} \tag{23}$$

$$\partial_\tau \mathcal{Y} = \left(\frac{\zeta}{2}\sigma_3 + \mathcal{U}\right)\mathcal{Y} \tag{24}$$

where

$$\mathcal{V}_0 = \frac{1}{2}\begin{pmatrix} -\theta_\infty - \frac{\tau}{4} + \frac{n(\tau)}{2\sqrt{2s}} & \left(\frac{\tau}{8\sqrt{2s}}\mathcal{E}(\tau) - \frac{\rho(\tau)}{2\sqrt{2s}}\right)e^{\tau/2} \\ -\left(\frac{\tau}{8\sqrt{2s}}\bar{\mathcal{E}}(\tau) + \frac{\bar{\rho}(\tau)}{2\sqrt{2s}}\right)e^{-\tau/2} & \theta_\infty + \frac{\tau}{4} - \frac{n(\tau)}{2\sqrt{2s}} \end{pmatrix} \tag{25}$$

$$\mathcal{V}_1 = \frac{1}{2}\begin{pmatrix} -\theta_\infty + \frac{\tau}{4} - \frac{n(\tau)}{2\sqrt{2s}} & \left(\frac{\tau}{8\sqrt{2s}}\mathcal{E}(\tau) + \frac{\rho(\tau)}{2\sqrt{2s}}\right)e^{\tau/2} \\ -\left(\frac{\tau}{8\sqrt{2s}}\bar{\mathcal{E}}(\tau) - \frac{\bar{\rho}(\tau)}{2\sqrt{2s}}\right)e^{-\tau/2} & \theta_\infty - \frac{\tau}{4} + \frac{n(\tau)}{2\sqrt{2s}} \end{pmatrix} \tag{26}$$

$$\mathcal{U} = \frac{1}{8\sqrt{2s}}\begin{pmatrix} 0 & \mathcal{E}(\tau)e^{\tau/2} \\ -\bar{\mathcal{E}}(\tau)e^{-\tau/2} & 0 \end{pmatrix}. \tag{27}$$

An alternative way of deriving equations (23) and (24) is to employ the change of variables (22) in equations (14) and (2). Notice that we do not introduce new notation for the new functions $\mathcal{E}(\tau)$, $\rho(\tau)$ and $n(\tau)$ which are, respectively, equal to the functions $\mathcal{E}(z)$, $\rho(z)$ and $n(z)$ above†. The asymptotic expansions (8) and (9) are all transformed under the change of variables (22). *One finds that, in the vicinity of the essential singularity at* $\zeta = \infty$, the function $\mathcal{Y}(\zeta, t)$ has the asymptotic expansion

$$\mathcal{Y}(\zeta, \tau) = \left(I + \frac{\mathcal{Y}_{-1}(\tau)}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^2}\right)\right)\exp\left(\left(\frac{\tau}{2}\zeta - \theta_\infty \ln\zeta\right)\sigma_3\right) \tag{28}$$

and in the vicinity of the regular singular points $a_{0,1} = 0, 1$, it has the asymptotic expansion

$$\mathcal{Y}(\zeta, \tau) \sim \sum_{n=0}^{\infty} \mathcal{Y}_n(\tau)(\zeta - a_r)^n (\zeta - a_r)^{(\theta_r/2)\sigma_3} P_r$$

$$\zeta \sim a_r \qquad r = 0, 1. \tag{29}$$

It can readily be shown that the compatibility condition $(\partial_\zeta\partial_\tau\mathcal{Y} = \partial_\tau\partial_\zeta\mathcal{Y})$ for the linear system (23) with (24) *is the following system of nonlinear ordinary differential equations* (which is system (1) in terms of the self-similar variable τ),

$$\tfrac{1}{2}(\tau\mathcal{E}(\tau))' = \rho(\tau) + \theta_\infty\mathcal{E}(\tau) \qquad \tfrac{1}{2}(\tau\bar{\mathcal{E}}(\tau))' = \bar{\rho}(\tau) - \theta_\infty\bar{\mathcal{E}}(\tau)$$

$$4\sqrt{2s}\rho'(\tau) = \mathcal{E}(\tau)n(\tau) \qquad 4\sqrt{2s}\bar{\rho}'(\tau) = \bar{\mathcal{E}}(\tau)n(\tau) \tag{30}$$

$$4\sqrt{2s}n'(\tau) + \tfrac{1}{2}(\mathcal{E}(\tau)\bar{\rho}(\tau) + \bar{\mathcal{E}}(\tau)\rho(\tau)) = 4s.$$

† Strictly speaking we should write $\bar{\mathcal{E}}(\tau) = \mathcal{E}(z)$ and so on.

This system is solvable at least locally which means that the linear system (23) with (24) is compatible. Because of this fact it is possible to construct a solvable RH problem by using the analytical properties with respect to ζ of the solutions of equation (23) (see e.g. [9, 10]).

On the other hand the existence of equation (24) means that the solutions of system (30) define isomonodromic deformations of system (23). As shown by Jimbo and Miwa [15], the isomonodromic deformations of system (23) with (24) can be described by solutions of $\mathbf{P}_5$ which is

$$\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \frac{1}{\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{(y-1)^2}{\tau^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{\tau}y + \frac{\delta y(y+1)}{y-1}. \tag{31}$$

The parameters α, β, γ and δ are connected with θ_r and θ_∞ (see [15]) such that

$$\begin{aligned} &\alpha = \tfrac{1}{4}(\theta_0 - \theta_1 + 2\theta_\infty)^2 \qquad \beta = -\tfrac{1}{4}(\theta_0 - \theta_1 - 2\theta_\infty)^2 \\ &\gamma = 1 - \theta_0 - \theta_1 \qquad \delta = -\tfrac{1}{2}. \end{aligned} \tag{32}$$

By comparing the matrices $\mathcal{V}_0$, $\mathcal{V}_1$ and $\mathcal{U}$ with the results of [15] concerning $\mathbf{P}_5$, we can find a parametrization of the solution of system (1) in terms of the solution of the fifth Painlevé equation. In this way we find

$$\mathcal{E}(\tau) = 2\sqrt{2s}u\left(\frac{2\tau(y-y')}{y-1} + y(\theta_0 - \theta_1 + 2\theta_\infty) - \theta_0 + \theta_1 + 2\theta_\infty\right)\frac{\mathrm{e}^{-\tau/2}}{\tau} \tag{33}$$

$$\bar{\mathcal{E}}(\tau) = -\frac{2\sqrt{2s}}{uy}\left(\frac{2\tau(y-y')}{y-1} + \theta_0 - \theta_1 - 2\theta_\infty - y(\theta_0 - \theta_1 + 2\theta_\infty)\right)\frac{\mathrm{e}^{\tau/2}}{\tau} \tag{34}$$

$$\begin{aligned} \rho(\tau) = \sqrt{2s}u\bigg(&\frac{\tau(y-y')(y+1)}{(y-1)^2} + \frac{2}{y-1}(\theta_0+\theta_1) + \frac{y}{2}(\theta_0 - \theta_1 + 2\theta_\infty) \\ &+ \frac{1}{2}(5\theta_0 + 3\theta_1 - 2\theta_\infty)\bigg)\mathrm{e}^{-\tau/2} \end{aligned} \tag{35}$$

$$\begin{aligned} \bar{\rho}(\tau) = -\frac{\sqrt{2s}}{u}\bigg(&\frac{\tau(y-y')(y+1)}{y(y-1)^2} + \frac{2}{y-1}(\theta_0+\theta_1) - \frac{1}{2y}(\theta_0 - \theta_1 - 2\theta_\infty) \\ &- \frac{1}{2}(\theta_0 - \theta_1 + 2\theta_\infty)\bigg)\mathrm{e}^{\tau/2} \end{aligned} \tag{36}$$

$$n(\tau) = 2\sqrt{2s}\left(\frac{\tau(y-y')}{(y-1)^2} + \frac{\theta_0+\theta_1}{y-1} + \frac{1}{2}(\theta_0+\theta_1) + \frac{\tau}{4}\right) \tag{37}$$

where u is the solution of

$$\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(u) = \frac{\tau}{2}\left(1 - \frac{y'}{y}\right) + \frac{y}{2}(\theta_0 - \theta_1) - \frac{1}{2}(\theta_0 - \theta_1 - 4\theta_\infty) - \frac{(y-1)^2}{4y}(\theta_0 - \theta_1 - 2\theta_\infty). \tag{38}$$

Notice that two integrals of motion of system (30) can easily be derived. By substituting the asymptotic expansion (29) into equation (23) we find that the eigenvalues of the matrices $\mathcal{V}_0$

and $\mathcal{V}_1$ are $\pm\theta_0$ and $\pm\theta_1$ respectively. It follows that $\det \mathcal{V}_r = -\theta_r{}^2$; $r = 0, 1$, are integrals of system (30), where

$$\theta_0^2 = \left(\theta_\infty + \frac{\tau}{4} - \frac{n(\tau)}{2\sqrt{2s}}\right)^2 - \frac{\tau^2}{128s}\mathcal{E}(\tau)\bar{\mathcal{E}}(\tau) + \frac{1}{8s}\rho(\tau)\bar{\rho}(\tau) - \frac{\tau}{32s}\left(\mathcal{E}(\tau)\bar{\rho}(\tau) - \bar{\mathcal{E}}(\tau)\rho(\tau)\right) \tag{39}$$

$$\theta_1^2 = \left(\theta_\infty - \frac{\tau}{4} + \frac{n(\tau)}{2\sqrt{2s}}\right)^2 - \frac{\tau^2}{128s}\mathcal{E}(\tau)\bar{\mathcal{E}}(\tau) + \frac{1}{8s}\rho(\tau)\bar{\rho}(\tau) + \frac{\tau}{32s}(\mathcal{E}(\tau)\bar{\rho}(\tau) - \bar{\mathcal{E}}(\tau)\rho(\tau)). \tag{40}$$

In the case of the real reduction,

$$\tau = \tau^* \qquad \bar{\mathcal{E}}(\tau) = \mathcal{E}^*(\bar{\tau}) = \mathcal{E}(\tau) \qquad \theta_0^2 = (\theta_0^*)^2 = \theta_1^2 \qquad \theta_\infty = 0 \tag{41}$$

we recover the particular case of these integrals obtained by Winternitz [4].

4. Similarity reduction to the second Painlevé equation

Another example of specifying the data of the RH problem is provided by a particular case of equation (12) for which

$$G_j(k) = \exp(Ak^3\sigma_3)G_j^{(0)}\exp(-Ak^3\sigma_3). \tag{42}$$

Here A is a new arbitrary parameter independent of ξ and η. By using the same arguments as in section 3 we can find the auxiliary equation the function $\Psi(\xi, \eta, k)$ must satisfy,

$$\Psi_k - 3Ak\Psi_\xi - (k/s)\Psi_\eta = 0. \tag{43}$$

The solutions of equation (43) have a self-similar form,

$$\Psi(\xi, \eta, k) = \Phi(\lambda, \tilde{z}) \qquad \lambda = \sqrt{2s\eta + k^2} \qquad \tilde{z} = \xi - 3As\eta. \tag{44}$$

In the same way as in section 3, i.e. by using equations (44), (8) and (11), we can construct the self-similar substitution for the nonlinear system (1),

$$E(\xi, \eta) = \epsilon(\tilde{z}) \qquad \varrho(\xi, \eta) = r(\tilde{z}) \qquad N(\xi, \eta) = \tilde{n}(\tilde{z}). \tag{45}$$

If in system (2) with (3) we change the variables ξ and η to the self-similar variable $\tilde{z}$ given by equation (44), then, after appropriate scale transformations, it takes the form

$$\partial_\varsigma Y = (\varsigma^2\sigma_3 + \varsigma V_1 + V_0)Y \tag{46}$$

$$\partial_{\tilde{\tau}} Y = (\tfrac{1}{2}\varsigma\sigma_3 + U)Y \tag{47}$$

where

$$Y(\varsigma, \tilde{\tau}) = \Psi(\xi, \eta, k) \qquad \varsigma = \sqrt[3]{3A}\lambda \qquad \tilde{\tau} = \frac{2}{\sqrt[3]{3A}}\tilde{z} \tag{48}$$

and

$$V_0 = \frac{1}{4s\sqrt[3]{3A}} \begin{pmatrix} \tilde{n}(\tilde{\tau}) & -r(\tilde{\tau}) \\ -\bar{r}(\tilde{\tau}) & -\tilde{n}(\tilde{\tau}) \end{pmatrix} \tag{49}$$

$$V_1 = 2U = \frac{\sqrt[3]{3A}}{2} \begin{pmatrix} 0 & \epsilon(\tilde{\tau}) \\ -\bar{\epsilon}(\tilde{\tau}) & 0 \end{pmatrix}. \tag{50}$$

Notice that the new functions $\epsilon(\tilde{\tau})$, $r(\tilde{\tau})$ and $\tilde{n}(\tilde{\tau})$ are equal to the old functions $\epsilon(\tilde{z})$, $r(\tilde{z})$ and $\tilde{n}(\tilde{z})$ respectively. In the vicinity of the essential singular point at $\varsigma = \infty$ the following asymptotic expansion is valid:

$$Y(\varsigma, \tilde{\tau}) \sim \left(I + \frac{Y_{-1}(\tilde{\tau})}{\varsigma} + \mathcal{O}\left(\frac{1}{\varsigma^2}\right)\right) \exp\left(\left(\frac{\varsigma^3}{3} + \frac{\varsigma\tilde{\tau}}{2} - \theta_\infty \ln\varsigma\right)\sigma_3\right). \tag{51}$$

The nonlinear system (1) expressed in terms of the self-similar variable $\tilde{\tau}$ (see equations (48)) now takes the form

$$\begin{aligned} &-2s(3A)^{2/3}\epsilon'(\tilde{\tau}) = r(\tilde{\tau}) \qquad -2s(3A)^{2/3}\bar{\epsilon}'(\tilde{\tau}) = \bar{r}(\tilde{\tau}) \\ &2r'(\tilde{\tau}) = \sqrt[3]{3A}\tilde{n}(\tilde{\tau})\epsilon(\tilde{\tau}) \qquad 2\bar{r}'(\tilde{\tau}) = \sqrt[3]{3A}\tilde{n}(\tilde{\tau})\bar{\epsilon}(\tilde{\tau}) \\ &\frac{2}{\sqrt[3]{3A}}\tilde{n}'(\tilde{\tau}) + \tfrac{1}{2}(\epsilon(\tilde{\tau})\bar{r}(\tilde{\tau}) + \bar{\epsilon}(\tilde{\tau})r(\tilde{\tau})) = 4s. \end{aligned} \tag{52}$$

Equations (52) are the compatibility condition for the linear system (46) with (47) or, in other words, equations for the isomonodromic deformations of equation (46). On the other hand, it was shown by Jimbo and Miwa [15] that these isomonodromic deformations are given by the second Painlevé equation ($\mathbf{P}_2$),

$$\frac{\mathrm{d}^2y}{\mathrm{d}\tilde{\tau}^2} = 2y^3 + \tilde{\tau}y + \nu. \tag{53}$$

Comparison of the matrices V_0, V_1 and U and the results of Jimbo and Miwa [15] for $\mathbf{P}_2$ leads to a parametrization of the solutions of system (46) in terms of the solutions of the second Painlevé equation such that

$$\epsilon(\tilde{\tau}) = \frac{2}{\sqrt[3]{3A}}\sqrt{F}\exp\left(\frac{\theta_\infty}{2}\int^{\tilde{\tau}}\frac{\mathrm{d}\tilde{\tau}}{F} + \phi_0\right) \tag{54}$$

$$\bar{\epsilon}(\tilde{\tau}) = \frac{4}{\sqrt[3]{3A}}\sqrt{F}\exp\left(-\frac{\theta_\infty}{2}\int^{\tilde{\tau}}\frac{\mathrm{d}\tilde{\tau}}{F} - \phi_0\right) \tag{55}$$

$$\tilde{n}(\tilde{\tau}) = 4s\sqrt[3]{3A}(y' - y^2) \tag{56}$$

$$r(\tilde{\tau}) = 4s\sqrt[3]{3A}y\sqrt{F}\exp\left(\frac{\theta_\infty}{2}\int^{\tilde{\tau}}\frac{\mathrm{d}\tilde{\tau}}{F} + \phi_0\right) \tag{57}$$

$$\bar{r}(\tilde{\tau}) = 8s\sqrt[3]{3A}\frac{\theta_\infty + yF}{\sqrt{F}}\exp\left(-\frac{\theta_\infty}{2}\int^{\tilde{\tau}}\frac{\mathrm{d}\tilde{\tau}}{F} - \phi_0\right) \tag{58}$$

where ϕ_0 is an arbitrary constant. Here F satisfies the equation

$$F'' = -2F^2 + \frac{(F')^2}{2F} - \frac{\theta_\infty^2}{2F} - \tilde{\tau}F \tag{59}$$

which is connected to (53) by the formulae

$$F = y' - y^2 - \frac{\tilde{\tau}}{2} \qquad y = -\frac{F' + \theta_\infty}{2F} \qquad \nu = \frac{1}{2} - \theta_\infty. \tag{60}$$

5. Asymptotics

Recall first that there is a physical context for the first reduction of system (1) (see introduction). The system of the Maxwell–Bloch equations,

$$\partial_\eta E = \varrho \qquad \partial_\xi \varrho = NE \qquad \partial_\xi N + \tfrac{1}{2}(E\varrho^* + E^*\varrho) = 0 \tag{61}$$

describes the interaction of an optical pulse with an active medium consisting of two-level atoms in the case of an infinitely narrow spectral line (see e.g. [16–18]). In this case ξ and η are real variables connected with the natural spacetime variables x and t,

$$\eta = \Omega x/c \qquad \xi = \Omega(t - x/c). \tag{62}$$

Here $\Omega = 2\pi n_0 \omega_0 \mu^2/\hbar$, n_0 is the density of atoms in the active medium, μ is the dipole matrix element of the two-level atom and ω_0 is the resonance frequency.

The Maxwell–Bloch system is derived by substituting into the Maxwell equations a linearly polarized plane wave with slowly varying amplitude and phase and moving to the right. In the framework of the inverse method as applied to system (61), x plays the role of the evolution variable and therefore the Cauchy problem for the Maxwell–Bloch system is naturally posed on the semi-axis $x \geqslant 0$.

The dynamical variables E and ϱ are slowly varying complex amplitudes of the electric field and polarization respectively. The real variable N is the population inversion of the excited atoms in the medium. It is evident that N varies within the interval $[-1, 1]$ for physically relevant solutions.

As mentioned in the introduction, an extension of the inverse scattering transform method [1] leads to a 'deformation' of system (61) such that

$$\partial_\eta E = \varrho \qquad \partial_\xi \varrho = NE \qquad \partial_\xi N + \tfrac{1}{2}(E\varrho^* + E^*\varrho) = 4s. \tag{63}$$

Although the derivation of this system given in [1] is entirely formal, the constant source term $4s > 0$ can be interpreted as an additional excitation of the active medium due to external sources. This is why the authors of [1] called the system (63) the 'Maxwell–Bloch system with a pumping' or the 'deformed Maxwell–Bloch system'. It is evident that a more detailed investigation of a possible origin for this interaction is needed for a physical realization of the deformed Maxwell–Bloch system.

The initial data of the Cauchy problem for a nonlinear partial differential equation usually have no similarity structure when a real physical process is described. In other words the similarity structure of the initial data corresponds to a very special initial state of the relevant physical system. Moreover, very often a similarity solution has a singularity at the initial point which means that the corresponding initial state of the physical system cannot be prepared.

Nevertheless, for a large class of soliton equations, it can be proven that solutions for rapidly decreasing initial data show similarity at late times. For integrable models, in particular, we have in mind here the schemes proposed in [19] and [20]. In this section, assuming that the approaches developed in [19, 20] can be applied to system (1), we shall derive the asymptotics of its similarity solutions.

In order to discuss the asymptotics of the solution of (63) in the case when it is reduced to $\mathbf{P}_5$, we need first to consider the choice of parameters in the relevant solution of $\mathbf{P}_5$. It is easy to verify that the reduction $\bar{\mathcal{E}} = \mathcal{E}^*$, $\bar{\rho} = \rho^*$ and $n = n^*$ in system (30) for $\tau = \tau^*$, $s > 0$ is provided by the specification of $\mathbf{P}_5$:

$$\theta_0 = \theta_1^* \qquad \theta_\infty = -\theta_\infty^* \qquad y = y^* < 0 \tag{64}$$

which defines a solution of (30) with six real parameters.

Notice that for $s < 0$, $\tau = -\tau^*$ the same reduction is achieved by the choice $\theta_0 = \theta_1 = \theta_1^*$, $\theta_\infty = -\theta_\infty^*$, $y = 1/y^*$. In a particular case of (64) for which

$$\theta_\infty = 0 \qquad \text{and} \qquad \theta_0 = \theta_1 = \theta_1^* \tag{65}$$

equations (33)–(37) are reduced to

$$\mathcal{E}(\tau) = 4\sqrt{2s}\frac{(y-y')}{\sqrt{y}(y-1)}\mathrm{e}^{\mathrm{i}\psi} \tag{66}$$

$$\rho(\tau) = \frac{2\sqrt{2s}}{\sqrt{y}}\left(\frac{\tau(y-y')(y+1)}{2(y-1)^2} + 2\theta_0\frac{y}{y-1}\right)\mathrm{e}^{\mathrm{i}\psi} \tag{67}$$

$$n(\tau) = 2\sqrt{2s}\left(\frac{\tau(y-y')}{(y-1)^2} + \theta_0\frac{y+1}{y-1} + \frac{\tau}{4}\right) \tag{68}$$

where ψ is an arbitrary real constant. Equations (66)–(68) define a solution of system (30) which depends on four real parameters. It is shown in [21] that, when the parameters of $\mathbf{P}_5$ are specified by equation (65), the asymptotic behaviour of the solution of $\mathbf{P}_5$ for $\tau \to \infty$ is given by the two-parameter asymptotic expansion

$$y = -1 + \frac{8a}{\sqrt{\tau}}\cos\omega + \mathcal{O}\left(\tau^{-1}\right) \tag{69}$$

where ($\omega = \frac{1}{2}\tau - 2a^2\ln\frac{\tau}{4} + \alpha$). Here $0 \leqslant a$ and α are real parameters. This solution is negative for $\tau \to \infty$ which is important for our approach. For the choice (65), the original dynamical variables take the form

$$E(\xi,\eta) = \eta^{1/2}\mathcal{E}(\tau) \qquad \varrho(\xi,\eta) = \eta^{-1/2}\rho(\tau) \qquad N(\xi,\eta) = \eta^{-1/2}n(\tau). \tag{70}$$

By substituting expansion (69) into (66)–(68) we find the following asymptotic expansions in terms of independent variables $x \to +\infty$ and t (see (62)):

$$E(x,t) = -2\mathrm{i}\mathrm{e}^{\mathrm{i}\psi}\left(\sqrt{\frac{2s\Omega}{c}x} - \frac{2a}{p\sqrt{2s}}\sin\omega\right) + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right) \tag{71}$$

$$\varrho(x,t) = \mathrm{i}N_0\mathrm{e}^{\mathrm{i}\psi}\cos\omega + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right) \tag{72}$$

$$N(x,t) = N_0\sin\omega + \mathcal{O}\left(\frac{1}{\sqrt{x}}\right). \tag{73}$$

Here $N_0 = 8asp$, and $p^2 = \xi/\sqrt{2s\eta} = \text{constant} \geqslant 0$. Using the physical condition that $|N| \leqslant 1$ one finds that the constant p can be chosen arbitrarily within the interval $|p| \leqslant 1/8as$, and thus asymptotics (71)–(73) are valid in the spacetime sector

$$x,t \to +\infty \qquad \frac{ct}{x} = 1 + p^2\sqrt{\frac{2s}{\Omega t}} + \mathrm{o}\left(\frac{1}{\sqrt{t}}\right). \tag{74}$$

We turn next to the appropriate asymptotic solutions in the case of reduction to $\mathbf{P}_2$.

It is evident from equations (54)–(58) that for $\tilde{\tau} = \tilde{\tau}^*$, $s > 0$ the reduction $\bar{\epsilon}(\tilde{\tau}) = \epsilon^*(\tilde{\tau})$, $\bar{r}(\tilde{\tau}) = r^*(\tilde{\tau})$, $\tilde{n}(\tilde{\tau}) = \tilde{n}^*(\tilde{\tau})$ can be realized if and only if $F(\tilde{\tau}) = F^*(\tilde{\tau}) > 0$, $\theta_\infty = -\theta^*_\infty = \mathrm{i}\alpha$ and $\phi_0 = \ln\sqrt{2} + \mathrm{i}\beta$. Now, it follows from equations (60) that, even though $y(\tilde{\tau})$ for real $F(\tilde{\tau})$ is complex, $\tilde{n}(\tilde{\tau})$ defined by equation (56) is real. As before $y(\tilde{\tau})$ should be chosen in such a way that $\tilde{n}(\tilde{\tau})$ varies within the interval $[-1, 1]$. There is a three-real-parameter family of similarity solutions which satisfy these requirements. This family is parametrized by the parameter β and by a two-real-parameter family of $\mathbf{P}_2$ solutions which for $\tilde{\tau} \to -\infty$ have the asymptotics

$$y = (1/2 - \alpha)/(-\tilde{\tau}) + \mathcal{O}(\tilde{\tau}^{-4}). \tag{75}$$

The second parameter is α and the third is the coefficient of an exponentially small term which is omitted in (75) (see [21]). By substituting (75) into (54)–(58) and (60), we finally arrive at the following asymptotic solution for the deformed Maxwell–Bloch system:

$$\epsilon(\tilde{\tau}) = \frac{2}{\sqrt[3]{3A}}(-\tilde{\tau})^{1/2-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta} + \mathcal{O}(\tilde{\tau}^{-2}) \tag{76}$$

$$r(\tilde{\tau}) = 4s\sqrt[3]{3A}(\tfrac{1}{2} - \mathrm{i}\alpha)(-\tilde{\tau})^{-1/2-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta} + \mathcal{O}(\tilde{\tau}^{-3}) \tag{77}$$

$$\tilde{n}(\tilde{\tau}) = s\sqrt[3]{3A}(1 + 4\alpha^2)/\tilde{\tau}^2 + \mathcal{O}(\tilde{\tau}^{-5}). \tag{78}$$

In the natural spacetime variables x and t,

$$\tilde{\tau} = \frac{2\Omega}{\sqrt[3]{3A}}\left(t - \frac{x}{v}\right) \tag{79}$$

where $v = c/(1 + 3As)$ is the phase velocity of the signal. Taking into account that $0 < v \leqslant c$, it is evident that $A \geqslant 0$.

The asymptotic solutions (76)–(78) in terms of the natural spacetime variables, are valid in the regions:

(i) for any $q < 1 + 3As$, $ct/x \leqslant q$ with $x \to +\infty$,

(ii) $t \to -\infty$ (recall that $x \geqslant 0$).

We can conclude that for both similarity reductions the electric field grows as $\sqrt{x}$ for $x \to +\infty$. It is interesting to notice that this behaviour for the electric field was found in [22] for a soliton solution of the deformed Maxwell–Bloch system.

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